Classical States via Decoherence

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Abstract

The initial states which minimize the predictability loss for a damped harmonic oscillator are identified as quasi-free states with a symmetry dictated by the environment's diffusion coefficients. For an isotropic diffusion in phase space, coherent states (or mixtures of coherent states) are selected as the most stable ones.

The emergence of classical reality from the underlying quantum description of the world is one of the most fascinating unsolved problems of present-day physics. Decoherence was proposed as a mechanism for the selection of classical (preferred) states: due to the interaction with an environment (external degrees of freedom) classical states are singled out as the most stable ones.

Zurek, Habib and Paz [1] invented a criterion, called "predictability sieve", for distinguishing the preferred set of states from the rest of the Hilbert space: classical states are characterized by the least increase in entropy. They addressed this problem in the context of the Caldeira-Leggett model (a particle moving in a potential and linearly coupled with a bath of harmonic oscillators, [1, 2, 3, 4]). They found out that coherent states are selected, via predictability sieve, as the most "classical" ones.

In contradistinction to this approach, we will study the problem of classicality in the framework of Lindblad's theory, in which the structure of the master equation is derived from very general assumptions concerning the mathematical description of the evolution of an open quantum system [5]. This strategy leads to general results (since Lindblad equations work for a large class of physically interesting systems [6]), and also avoids any problems related to the non-preservation

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of the positivity of the reduced density matrix, which was perceived by some authors [7] as an inconvenient of the Caldeira-Leggett type of evolution. We will identify the states with classical behavior as the states which minimize the rate of entropy increase. We will apply this criterion in an identical fashion for both pure and mixed initial states, using a formalism that doesn't require one to discriminate between them.

Lindblad's result shows that a general form for the generator of a completely positive dynamical semigroup in the Schrödinger picture is [5]

$$L(\rho) = -\frac{i}{\hbar}[H, \rho] + \frac{1}{2\hbar}\sum_j ([V_j\rho, V_j^+] + [V_j, \rho V_j^+]), \tag{1}$$

where for the operators V_j, we will take [6] $V_j = a_j p + b_j q$, where $j = 1, 2$ while a_j and b_j are complex numbers. For the Hamiltonian H of the system we will consider [6] a second-order polynomial operator in coordinate and position written as a sum of a harmonic-oscillator part and a mixed term

$$H = \frac{m\omega^2}{2}q^2 + \frac{p^2}{2m} + \frac{\mu}{2}(pq + qp). \tag{2}$$

We denote the diffusion coefficients by

$$D_{qq} = \frac{\hbar}{2}\sum_{j=1}^{2}|a_j|^2, \quad D_{pp} = \frac{\hbar}{2}\sum_{j=1}^{2}|b_j|^2, \quad D_{pq} = D_{qp} = -\frac{\hbar}{2}Re\sum_{j=1}^{2}a_j^* b_j, \tag{3}$$

and the friction constant is

$$\lambda = -Im\sum_{j=1}^{2}a_j^* b_j. \tag{4}$$

It is easy to show [6] that the following inequality must be satisfied

$$D_{pp}D_{qq} - D_{pq}^2 \geq \frac{\lambda^2\hbar^2}{4}. \tag{5}$$

With the above notation, the master equation which governs the evolution of the system takes the form

$$
\begin{aligned}
\frac{d\rho(t)}{dt} &= -\frac{i}{\hbar}[H, \rho(t)] - \frac{i\lambda}{\hbar}([q, \rho(t)p] - [p, \rho(t)q]) \\
&\quad -\frac{D_{qq}}{\hbar^2}[p, [p, \rho(t)]] - \frac{D_{pp}}{\hbar^2}[q, [q, \rho(t)]] + \frac{2D_{pq}}{\hbar^2}[p, [q, \rho(t)]].
\end{aligned} \tag{6}
$$

For the correlations between two operators C and C' we will use the definition

$$\sigma_{C,C'}(t) = Tr\left(\rho(t)\frac{CC' + C'C}{2}\right) - \sigma_C(t)\sigma_{C'}(t), \tag{7}$$

where $\sigma_C(t) = Tr(\rho(t)C)$ and $\sigma_{C'}(t) = Tr(\rho(t)C')$. $\Sigma(t)$ will denote the dispersion matrix

$$\Sigma(t) = \begin{pmatrix} m\omega\sigma_{qq}(t) & \sigma_{pq}(t) \\ \sigma_{pq}(t) & \frac{\sigma_{pp}(t)}{m\omega} \end{pmatrix}. \tag{8}$$

With the notations above, Heisenberg's inequality reads $det\Sigma(t) \geq \frac{\hbar^2}{4}$. For $\Sigma(t)$ the time-evolution is known [6]

$$\frac{d\Sigma(t)}{dt} = Y\Sigma(t) + \Sigma(t)Y^T + 2\mathcal{D}, \qquad (9)$$

where

$$Y = \begin{pmatrix} -(\lambda - \mu) & \omega \\ -\omega & -(\lambda + \mu) \end{pmatrix},$$

Y^T is the transposed matrix of Y and $\mathcal{D}$ is a 2×2 matrix with elements $\mathcal{D}_{qq} = m\omega D_{qq}$, $\mathcal{D}_{pq} = \mathcal{D}_{qp} = D_{qp}$ and $\mathcal{D}_{pp} = \frac{D_{pp}}{m\omega}$.

For the case of the damped harmonic oscillator, it is known that a certain class of states (quasi-free states [8]) has the property of invariance under the action of the quantum semigroup. This can be seen by writing the Fokker-Planck equation which corresponds to (6) in the form [9]

$$\frac{\partial f_W(x_1, x_2, t)}{\partial t} = \sum_{i,j=1}^{2} Y_{ij} \frac{\partial}{\partial x_i}(x_j f_W(x_1, x_2, t)) + \frac{1}{2\hbar} \sum_{i,j=1}^{2} \mathcal{D}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f_W(x_1, x_2, t), \qquad (10)$$

where f_W is the Wigner function of a quasi-free state

$$f_W(x_1, x_2, t) = [(2\pi)^2 det\Sigma(t)]^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}X^T(t)\Sigma^{-1}X(t) \right\}, \qquad (11)$$

and $X(t) = \begin{pmatrix} x_1 - \sigma_q(t) \\ x_2 - \sigma_p(t) \end{pmatrix}$. It is now easy to verify that the Gaussian Wigner function (11), with the time-dependence of the mean values $\sigma_p(t)$, $\sigma_q(t)$ and of dispersions $\sigma_{qq}(t)$, $\sigma_{pp}(t)$, $\sigma_{pq}(t)$ given by equations (3.26 - 3.27) from [6], is a solution of the equation (10). Thus, quasi-free states are preserved during the evolution of the system.

Starting with such a state, we are interested to calculate the rate of linear entropy variation. The initial states with the lowest rate of linear entropy increase will be identified as the most stable (*i.e.* the most classical-like) states.

The linear entropy is a convenient measure of the purity of a quantum state and is defined by

$$s(\rho) = 1 - Tr(\rho^2). \qquad (12)$$

For a quasi-free state we have

$$s(\rho) = 1 - \frac{1}{A}. \qquad (13)$$

where $A(t)$ is the "area" in phase space, $A(t) = \frac{2}{\hbar}\sqrt{det\Sigma(t)}$. The condition for a quasi-free state to be a pure one is $A(t) = 1$, that is $det\Sigma(t) = \frac{\hbar^2}{4}$ (the equality case in the Heisenberg relation). The time-derivative of $s(t)$ can be calculated by making use of the following relation

$$\frac{d(\ln det\Sigma(t))}{dt} = Tr\left(\frac{d\Sigma(t)}{dt}\Sigma^{-1}(t)\right), \qquad (14)$$

where

$$\Sigma^{-1}(t) = \frac{1}{det\Sigma(t)} \begin{pmatrix} \frac{\sigma_{pp}(t)}{m\omega} & -\sigma_{pq}(t) \\ -\sigma_{pq}(t) & m\omega\sigma_{qq}(t) \end{pmatrix}. \qquad (15)$$

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Then we obtain from (9) and (14) that

$$\frac{dA(t)}{dt} = \frac{2}{\hbar}\sqrt{det\Sigma(t)}(TrY + Tr(\mathcal{D}\Sigma^{-1}(t))). \tag{16}$$

But the rate of the linear entropy increase is given by

$$\frac{ds(t)}{dt} = \frac{1}{A(t)^2}\frac{dA(t)}{dt}, \tag{17}$$

so the behavior of the rate of linear entropy increase is given entirely by the time derivative (16) of the area $A(t)$ for all states (including pure states). In the following we shall find the squeezing parameter of the initial state for which the rate of increase of $A(t)$ is minimized.

Let us notice, for the beginning, that the positive 2×2 matrix $\Sigma(t)$ can be diagonalized, at any particular instant t, in the form (see also [12, 13])

$$\Sigma(t) = \frac{\hbar A(t)}{2}O^T(t)\begin{pmatrix} \aleph^2(t) & 0 \\ 0 & \aleph^{-2}(t) \end{pmatrix}O(t). \tag{18}$$

where $\aleph(t)$ is a real positive number (the squeezing parameter) $A(t)$ is the area occupied by the system in phase space and $O(t)$ is an orthogonal symplectic matrix for which we will employ the usual form

$$O(t) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}.$$

A similar formula holds for the $\mathcal{D}$-matrix

$$\mathcal{D} = \frac{\hbar\Delta}{2}O_D^T\begin{pmatrix} d^2 & 0 \\ 0 & d^{-2} \end{pmatrix}O_D, \tag{19}$$

with

$$O_D = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{20}$$

Here, Δ is a parameter controlling the intensity of diffusion, d characterizes the degree of anisotropy and φ is the rotation angle.

Now, (16) and (17) imply

$$\left.\frac{ds}{dt}\right|_{t=0} = \frac{1}{A(0)}\left[-2\lambda + Tr\left(\Sigma^{-1}(0)\mathcal{D}\right)\right]. \tag{21}$$

This result is even more general when the system is in a pure state at t=0 (so that $A(0) = 1$), in the sense that it does not depend on the kind of initial states we are starting with (quasi-free states or not). Indeed, in [6] it was shown that

$$\frac{dTr(\rho^2)}{dt} = 2Tr(\rho L(\rho)) = \frac{2}{\hbar}\sum_j(Tr(\rho V_j\rho V_j^*) - Tr(\rho^2 V_j^*V_j)) \geq 0. \tag{22}$$

For a pure state $\rho^2 = \rho$ and $\rho O\rho = Tr(\rho O)\rho$ for any selfadjoint operator O. Then

$$\frac{dTr(\rho^2)}{dt} = \frac{2}{\hbar}\sum_j(|Tr(\rho V_j)|^2 - Tr(\rho V_j^*V_j)) \geq 0. \tag{23}$$

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We have $Tr(\rho V_j) = a_j\sigma_p + b_j\sigma_q$. Hence we get

$$
\begin{aligned}
\left.\frac{d}{dt}(1 - Tr(\rho^2))\right|_{t=0} &= -2\lambda + \frac{4}{\hbar^2}(D_{qq}\sigma_{pp}(0) + D_{qq}\sigma_{pp}(0) - 2D_{pq}\sigma_{pq}(0)) \\
&= -2\lambda + Tr\left(\Sigma^{-1}(0)\mathcal{D}\right),
\end{aligned}
$$

which has the same form as the rate of linear entropy increase calculated before (see (21)) for pure ($A(0) = 1$) initial quasi-free states.

We are interested in finding the states which produce the least increase of the phase-space area at the initial moment $t = 0$. By minimizing the expression

$$
\begin{aligned}
-2\lambda + Tr\left(\Sigma^{-1}(0)\mathcal{D}\right) &= -2\lambda + \frac{\Delta}{A(0)}\{\cos^2(\theta(0) - \varphi)[\aleph^2(0)d^{-2} + \aleph^{-2}(0)d^2] \\
&\quad + \sin^2(\theta(0) - \varphi)[\aleph^2(0)d^2 + \aleph^{-2}(0)d^{-2}]\},
\end{aligned}
$$

with respect to $\aleph(0)$ and $\theta(0)$, one gets

$$
min\left[\frac{ds}{dt}\right]\bigg|_{t=0} = 2\frac{\Delta - A(0)\lambda}{A(0)^2}. \tag{24}
$$

corresponding to

$$
\begin{aligned}
\aleph^*(0) &= d, \\
\theta^*(0) &= \varphi.
\end{aligned} \tag{25}
$$

So, in the general case of an anisotropic diffusion, the minimum variation of the area in phase space is obtained when the squeezing parameter of the state equals d, the degree of anisotropy of the diffusion, and the characteristic rotation angle of the correlation and diffusion matrices are equal. For an isotropic diffusion, $\mathcal{D}_{pp} = \mathcal{D}_{qq}$, $\mathcal{D}_{pq} = 0$, *i.e.* d=1 (and the rotation angle vanishes trivially from all the relations, since now the system has rotation symmetry in phase space); we obtain $\aleph^*(0) = 1$. This case corresponds to many models of dissipation, especially from quantum optics (see [11]); the same result was obtained by Zurek, Habib and Paz [1] in the context of the Caldeira-Leggett environment model. In other words, a phase-space isotropic environment favors a symmetric state, while an anisotropic environment selects states with the same degree of anisotropy.

When the initial state is pure,

$$
min\left[\frac{ds}{dt}\right]\bigg|_{t=0} = 2\left(\Delta - \lambda\right) \geq 0, \tag{26}
$$

where the last inequality comes from (5) and expresses the fact that the linear entropy of pure states in an environment always increases, because the pure initial states become more and more mixed.

Our result shows that the values of $\aleph^*(0)$ and $\theta^*(0)$ are independent of the overall magnitude Δ of the diffusion. They are also independent of $A(0)$. Thus, the same degree of squeezing is singled out, irrespective of the purity of the initial state, thus confirming previous insights [1, 4] regarding the structure of the mixed preferred states: they can be seen as the thermalization of the selected pure states.

We conclude by emphasizing the main results of this paper. In general, the pure or mixed state which produces the minimum rate of increase in the area occupied by the system in phase–space is a quasi-free state which has the same symmetry as that induced on the evolution in phase-space by the diffusion coefficients. For isotropic phase-space diffusion, the selected pure states are the coherent states.

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